Journal of Engineering Mathematics (2005) 52: 251–264
DOI 10.1007/s10665-004-5662-9

Compression and shear of a layer of granular material

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Received 20 May 2004; accepted in revised form 27 October 2004

Abstract. A classical problem in metal plasticity is the compression of a block of material between rigid platens. The corresponding problem for a layer of granular material that conforms to the Coulomb-Mohr yield condition and the double-shearing theory for the velocity field has also been solved. A layer of granular material between rough rigid plates that is subjected to both compression and shearing forces is considered. Analytical solutions are obtained for the stress and velocity fields in the layer. The known solutions for steady simple shear and pure compression are recovered as special cases. Yield loads are determined for combined compression and shear in the case of Coulomb friction boundary conditions. Numerical results which describe the stress and velocity fields in terms of the normal and shear forces on the layer at yield are presented for the case in which the surfaces of the platens are perfectly rough. Post-yield behaviour is briefly considered.

Key words: combined loading, compression, geological faults, granular material, shear

1. Introduction

The compression in plane strain of a rectangular block of ideal plastic-rigid material between rough rigid platens is a classical problem in metal plasticity that is described in the standard texts such as [1,2]. For a block that is long in comparison to its thickness there is an analytical solution, also classical, due to Prandtl [3]. The corresponding problem for a layer of granular material, assuming the Coulomb-Mohr failure condition, was studied by Hartmann [4], who obtained the solution for the stress field, and by Marshall [5], who extended Hartmann's stress solution and also determined the velocity field, on the basis of the non-dilatant double-shearing theory proposed by Spencer [6–8]. The problem is a model, for example, for a raft supporting a building resting on a layer of soil covering a rigid substrate, and also has geophysical applications with reference to geological faults.

Another problem in granular material mechanics that has been the subject of extensive study is that of simple shear of a layer of granular material. In principle this is a straightforward problem involving homogeneous stress and velocity fields, and is basic for the definition and measurement of the material parameters that characterize granular materials. Spencer [8,9] considered the simple shear of a layer of granular material using the non-dilatant double-shearing theory, and showed that there is a time-independent steady stress solution in which the material flows uniformly under constant shear stress, but that this solution is unstable (in the remainder of this paper we shall refer to this solution as the 'steady simple shearing solution'). It was also shown in [9] that there is an unsteady solution in which stress is time-dependent and uniform shear flow occurs in conjunction with decreasing shear stress.

However, there seems to be no systematic study of the case in which a granular layer between rough plates is subjected to both compression and shear. This problem is also of interest in modelling geological faults (see, for example, [10,11]). The main purpose of this paper is to fill this gap by giving an analysis of the problem of compression and shear of a granular layer between rough platens, using the Coulomb condition for cohesionless material and the double-shearing theory for the velocity field.

The basic equations are described in Section 2. In Section 3 we derive an exact solution of these equations for a layer of granular material. In Sections 4 and 5 this solution is applied to the problem of a layer of granular material confined between rough rigid platens that exert both normal and tangential forces on the granular layer; Section 4 develops the solution for the stress field and the velocity field is found in Section 5. In Section 6 we present some numerical results and discussion.

2. General theory

All quantities are referred to a fixed system of rectangular Cartesian coordinates $Oxyz$. The components of the stress tensor σ are denoted as

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}, \tag{2.1}$$

and the components of the velocity $\mathbf{v}$ by (u, v, w). We consider plane strain in the (x, y) planes, so that $w = 0$, u and v are functions of x and y, and the relevant stress components are σ_{xx}, σ_{xy}, and σ_{yy}, all of which depend only on x and y. We write

$$p = -\frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right), \quad q = \left\{\frac{1}{4}\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \sigma_{xy}^2\right\}^{\frac{1}{2}}, \quad q \geq 0, \tag{2.2}$$

so that p and q are stress invariants that represent the mean in-plane hydrostatic pressure and the maximum shear stress, respectively. The stress angle ψ is defined by

$$\tan 2\psi = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}, \tag{2.3}$$

and is the angle that the principal stress axis associated with the algebraically greater principal stress makes with the x-axis (tensile stress is taken to be positive). Then the relevant stress components can be expressed as

$$\sigma_{xx} = -p + q\cos 2\psi, \quad \sigma_{yy} = -p - q\cos 2\psi, \quad \sigma_{xy} = q\sin 2\psi. \tag{2.4}$$

In soil mechanics terminology, the case $\cos 2\psi > 0$ corresponds to passive lateral pressure and $\cos 2\psi < 0$ corresponds to active lateral pressure.

The material is assumed to conform to the Coulomb–Mohr yield condition for cohesionless material

$$q \leq p \sin\phi, \tag{2.5}$$

where ϕ is the angle of internal friction and (2.5) holds as an equality whenever the material is undergoing deformation, in which case

$$\sigma_{xx} = q\left(\cos 2\psi - \operatorname{co}\sec\phi\right), \quad \sigma_{yy} = -q\left(\cos 2\psi + \operatorname{co}\sec\phi\right), \quad \sigma_{xy} = q\sin 2\psi. \tag{2.6}$$

The equations of equilibrium are, in plane strain and neglecting body forces

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}=0,$$
$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0. \tag{2.7}$$

In quasi-static flows, when inertia terms are neglected, the Coulomb–Mohr condition (2.5) (as an equality) and the equilibrium equations can be expressed as a pair of first-order partial differential equations for q and ψ, as follows

$$(\cos 2\psi-\operatorname{co}\sec\phi)\frac{\partial q}{\partial x}+\sin 2\psi\frac{\partial q}{\partial y}-2q\sin 2\psi\frac{\partial \psi}{\partial x}+2q\cos 2\psi\frac{\partial \psi}{\partial y}=0,$$
$$\sin 2\psi\frac{\partial q}{\partial x}-(\cos 2\psi+\operatorname{co}\sec\phi)\frac{\partial q}{\partial y}+2q\cos 2\psi\frac{\partial \psi}{\partial x}+2q\sin 2\psi\frac{\partial \psi}{\partial y}=0. \tag{2.8}$$

These equations are hyperbolic, with characteristic curves (termed α-and β-lines) given by

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\tan(\psi\pm\frac{\phi}{2}\pm\frac{\pi}{4}) \tag{2.9}$$

where the lower and upper signs refer to the α-lines and β-lines respectively.

These equations for the stress field are quite generally accepted. To complete the material description it is necessary to specify a 'flow rule' that relates the stress to the deformation. The choice of an appropriate flow rule is still a matter of debate and has been discussed extensively in the literature. We adopt the 'double-shearing' theory for deformation of granular material. An accessible formulation of this theory, with references to earlier work, is given in Spencer [8]; other recent studies that make comparisons between the double-shearing theory and some alternative theories are by Gremaud [12], Alexandrov [13] and Zhu *et al.* [14]. According to the double-shearing theory, in plane strain the velocity components u and v satisfy

$$\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos 2\psi-\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\sin 2\psi+\sin\phi\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}+2\Omega\right)=0, \tag{2.10}$$

where

$$\Omega=\dot{\psi}=\frac{\partial \psi}{\partial t}+u\frac{\partial \psi}{\partial x}+v\frac{\partial \psi}{\partial y}, \tag{2.11}$$

is the spin of the principal stress axes through a generic particle. Accordingly, through Ω, the deformation depends on the stress-rate as well as on the stress. Equation (2.11) expresses the assumption that the deformation takes place by shearing on the two families of surfaces on which the Coulomb critical shear stress is mobilized. If in addition the deformation is isochoric, then

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.12}$$

Although in many applications (2.12) is a good approximation, it is argued in Section 5 that it may not be appropriate in the case considered there. There are two main ways in which the double-shearing theory may be modified to include an element of compressibility. The first was proposed by Spencer and Kingston [15]; in this extended theory (2.10) and (2.11) are unchanged, but (2.12) is replaced by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\Delta, \tag{2.13}$$

where the dilatation-rate Δ is a scalar which in general is expected to be complicated function of the deformation and stress history, but for the purposes of this paper does not need to be specified. The second and better-known way to incorporate dilatancy in the theory is due to Mehrabadi and Cowin [16,17] and is based on an assumption that shear of the material is accompanied by an expansion in the direction normal to the shear plane. In practice this expansion-rate decreases as the shear deformation progresses, and so this mechanism is perhaps most important in the initial stages of deformation. The Spencer and Kingston mechanism attempts to incorporate non-directional volumetric effects, such as absorption of fluid, or crushing or uniform compaction of grains under applied pressure. The Mehrabadi and Cowin mechanism describes the expansion normal to the shear plane that arises in shear of initially compacted granular materials as grains overtake their neighbours. This effect is most marked in the initial stages of deformation, and the rate of the normal expansion decays as the flow becomes fully developed or approaches a critical state. The two mechanisms are not exclusive, and there is no reason why they should not both be included in a complete theory that incorporates dilatancy.

3. An exact solution for a layer of granular material

In this section we derive an exact solution of the equations of Section 2, which will be applied to the analysis of the stress and deformation in a granular layer. We choose the coordinate system Oxy to be such that the x-axis is parallel to the layer and y measures distance through the layer thickness, as shown in Figure 1. Then the mid-plane of the layer is $y=0$, its surfaces are $y=\pm h$, and its ends are $x=\pm L$.

The classical solutions of Prandtl [3] and Marshall [5] suggest that we look for stress solutions of the form

$$\psi=\psi(y), \quad q=q(x,y). \tag{3.1}$$

By substituting (3.1) in the equilibrium equations (2.8), we obtain

$$\begin{aligned} &(\cos 2\psi-\operatorname{co}\sec\phi)\frac{\partial q}{\partial x}+\sin 2\psi\frac{\partial q}{\partial y}+2q\cos 2\psi\frac{\mathrm{d}\psi}{\mathrm{d}y}=0,\\ &\sin 2\psi\frac{\partial q}{\partial x}-(\cos 2\psi+\operatorname{co}\sec\phi)\frac{\partial q}{\partial y}+2q\sin 2\psi\frac{\mathrm{d}\psi}{\mathrm{d}y}=0. \end{aligned} \tag{3.2}$$

Hence, by dividing each of (3.2) by q and solving for $\partial(\log q)/\partial x$ and $\partial(\log q)/\partial y$ it follows that

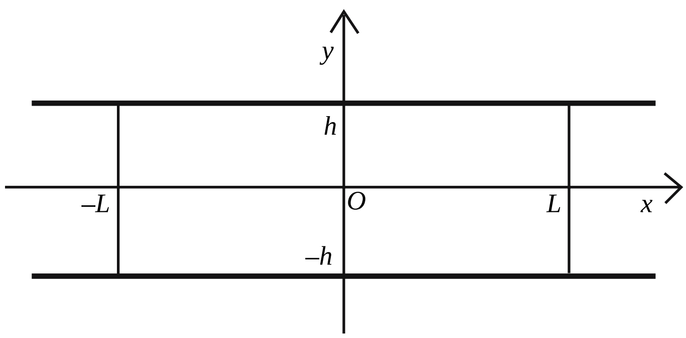

Figure 1. Compression and shear of a layer of granular material between rough rigid plates.

$$\cos\phi\cot\phi\frac{\partial}{\partial x}(\log q) = 2(\sin\phi + \cos 2\psi)\frac{d\psi}{dy},$$
$$\cos\phi\cot\phi\frac{\partial}{\partial y}(\log q) = 2\sin 2\psi\frac{d\psi}{dy}. \tag{3.3}$$

Since ψ is independent of x, we have $\partial^2(\log q)/\partial x\partial y = 0$, and therefore

$$\frac{d}{dy}\left\{2(\sin\phi + \cos 2\psi)\frac{d\psi}{dy}\right\} = 0 \tag{3.4}$$

which gives, on integration

$$2\psi\sin\phi + \sin 2\psi = a + b\frac{y}{h}, \tag{3.5}$$

where a and b are integration constants. This determines ψ implicitly as a function of y.

It also follows from (3.3_1) and (3.5) that

$$\cos\phi\cot\phi\log q = b\frac{x}{h} + g(y)$$

and by substituting this back in (3.3_2) we find

$$\frac{d}{dy}g(y) = 2\sin 2\psi\frac{d\psi}{dy}$$

and thus

$$g(y) = -\cos 2\psi + c, \tag{3.6}$$

where c is constant. Therefore,

$$\log q = (b\frac{x}{h} - \cos 2\psi + c)\tan\phi\sec\phi$$

or

$$q = C\exp\left\{(b\frac{x}{h} - \cos 2\psi)\tan\phi\sec\phi\right\}, \tag{3.7}$$

where $C = \exp(c\tan\phi\sec\phi)$. Thus the stress solution is determined to within the three constants a, b, and C. For the case $a = 0$, (3.5) and (3.7) reduce to the solution for pure compression of a granular layer between rough plates that was derived by Hartmann [4] and also discussed by Marshall [5]. The inclusion of the term a in (3.5) allows consideration of a wider class of solutions.

From (2.9) it follows that α- and β-lines corresponding to (3.5) are given parametrically by the equations

$$\frac{x - x_\alpha}{h} = \frac{1}{b}(\cos 2\psi - 2\psi\cos\phi), \quad \frac{y}{h} = -\frac{a}{b} + \frac{1}{b}(\sin 2\psi + 2\psi\sin\phi), \quad (\alpha\text{-lines})$$
$$\frac{x - x_\beta}{h} = \frac{1}{b}(\cos 2\psi + 2\psi\cos\phi), \quad \frac{y}{h} = -\frac{a}{b} + \frac{1}{b}(\sin 2\psi + 2\psi\sin\phi), \quad (\beta\text{-lines}), \tag{3.8}$$

where x_α is constant on a given α-line and x_β is constant on a given β-line. Hence the α- and β-lines are two families of cycloids. Each α-line is generated by a point on the circumference of a circle of radius h/b rolling on a line with slope $-\tan\phi$, and each β-line is generated by a point on the circumference of a similar circle rolling on a line with slope $\tan\phi$.

The steady simple shearing solution corresponds to the case $b = 0$. In this event it follows from (3.5) that ψ is constant, and the requirement that $\sigma_{xy} = \sigma_{yy}\tan\phi$ determines $\psi = \psi_0 = \frac{1}{2}\phi + \frac{1}{4}\pi$.

For the associated velocity field we consider separately the cases of incompressible and compressible material, as follows:

(*i*) We consider that the material is incompressible and suppose that the layer of granular material is confined between rigid platens at $y=\pm h$ that translate inwards with components of velocity in the y-direction of magnitude U, so that U is the speed of the surface $y=h$, and therefore $U=-\mathrm{d}h/\mathrm{d}t$. We seek solutions of (2.10) and (2.12) of the form

$$u=U\frac{x}{h}+\Phi(y), \quad v=-U\frac{y}{h}. \tag{3.9}$$

Then the isochoric condition (2.12) is identically satisfied, and (2.10) and (2.11) give

$$\frac{\mathrm{d}\Phi}{\mathrm{d}y}(\cos 2\psi+\sin\phi)-2\frac{U}{h}\sin 2\psi+2\sin\phi\left(\frac{\partial\psi}{\partial t}-U\frac{y}{h}\frac{\mathrm{d}\psi}{\mathrm{d}y}\right)=0. \tag{3.10}$$

Since h depends on t, therefore ψ depends on both y and t, and

$$\frac{\partial\psi}{\partial t}=\frac{\mathrm{d}h}{\mathrm{d}t}\frac{\partial\psi}{\partial h}=-U\frac{\partial\psi}{\partial h}=\frac{Uby}{2h^2(\sin\phi+\cos 2\psi)},$$
$$\frac{\mathrm{d}\psi}{\mathrm{d}y}=\frac{b}{h}\frac{1}{2(\sin\phi+\cos 2\psi)},$$

and hence from (2.11) in this case $\Omega=0$ and so (3.10) reduces to

$$\frac{\mathrm{d}\Phi}{\mathrm{d}y}(\cos 2\psi+\sin\phi)-2\frac{U}{h}\sin 2\psi=0. \tag{3.11}$$

Then, provided that $b\neq 0$, it follows from (3.5) that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}y}(\cos 2\psi+\sin\phi)=\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}y}(\cos 2\psi+\sin\phi)=\frac{b}{2h}\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}.$$

Hence (3.11) becomes

$$b\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}-4U\sin 2\psi=0, \tag{3.12}$$

and it follows that

$$b\Phi=-2U\cos 2\psi+bU_0,$$

where U_0 is a further integration constant that represents a rigid body translation in the x-direction. Hence from (3.9) the associated velocity field is given as

$$u=U\left(\frac{x}{h}-\frac{2}{b}\cos 2\psi\right)+U_0, \quad v=-U\frac{y}{h}. \tag{3.13}$$

In the case $b=0$, then ψ is constant and $U=0$, and the solution reduces to the steady simple shearing motion

$$u=\gamma y, \quad v=0.$$

(*ii*) We suppose that the material is confined between platens as in case (*i*) but that it is capable of undergoing compression. We seek solutions in which the material does not undergo stretching in the x-direction so that there is no slip between the layer and the platens, In order to satisfy this boundary condition it is natural to use the Spencer and Kingston theory [15], with the governing equations (2.10), (2.11) and (2.13). We assume that both of the velocity components u and v depend only on y, of the form

$$u = u(y), \quad v = -U\frac{y}{h}. \tag{3.14}$$

Then $\Omega = 0$ as in case (*i*) and (2.10) and (2.13) become

$$\frac{du}{dy}(\cos 2\psi + \sin\phi) - \frac{U}{h}\sin 2\psi = 0, \quad -\frac{U}{h} = \Delta. \tag{3.15}$$

Hence

$$\frac{du}{dy} = \frac{U}{h}\frac{\sin 2\psi}{(\cos 2\psi + \sin\phi)}, \tag{3.16}$$

and, from (3.5)

$$\frac{du}{d\psi} = 2\frac{U}{b}\sin 2\psi \tag{3.17}$$

and therefore the velocity field is

$$u = -\frac{U}{b}\cos 2\psi + U_0, \quad v = -U\frac{y}{h}. \tag{3.18}$$

The limiting case of steady simple shearing is an isochoric deformation, and so the response to volume changes is not relevant in this case.

4. Application to compression and shear of a granular layer. Stress field

We now apply the solution developed in Section 3 to a layer of granular material occupying the region $-L \le x \le L$, $-h \le y \le h$, subjected to compression and shear by normal forces N and shear forces S applied through rigid parallel platens at $y = \pm h$. We look for solutions in which the field of α- and β-lines has the form illustrated in Figure 2. For negative values of x, in the region $OA'D'C'B'O$, we assume that the stress is given by (3.5) and (3.7), thus

$$\begin{aligned} &2\psi\sin\phi + \sin 2\psi = a + b\frac{y}{h}, \\ &q = C\exp\left\{(b\frac{x}{h} - \cos 2\psi)\tan\phi\sec\phi\right\}, \end{aligned} \tag{4.1}$$

and the α- and β-lines have the parametric form (3.8), namely

$$\begin{aligned} &\tfrac{x-x_\alpha}{h} = \tfrac{1}{b}(\cos 2\psi - 2\psi\cos\phi),\ \tfrac{y}{h} = -\tfrac{a}{b} + \tfrac{1}{b}(\sin 2\psi + 2\psi\sin\phi),\ (\alpha\text{-lines}) \\ &\tfrac{x-x_\beta}{h} = \tfrac{1}{b}(\cos 2\psi + 2\psi\cos\phi),\ \tfrac{y}{h} = -\tfrac{a}{b} + \tfrac{1}{b}(\sin 2\psi + 2\psi\sin\phi),\ (\beta\text{-lines}). \end{aligned} \tag{4.2}$$

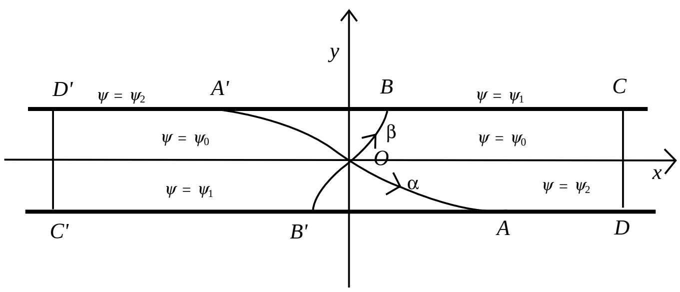

Figure 2. Schematic of slip-line field for compression and shear between rough rigid plates.

For positive values of x, we assume a similar solution, but with b replaced by $-b$, thus in $OADCBO$

$$2\psi \sin\phi + \sin 2\psi = a - b\frac{y}{h},$$
$$q = C \exp\left\{-(b\frac{x}{h} + \cos 2\psi)\tan\phi \sec\phi\right\}, \tag{4.3}$$

and the α- and β-lines have the parametric equations

$$\begin{aligned} \tfrac{x-x_\alpha}{h} &= -\tfrac{1}{b}(\cos 2\psi - 2\psi\cos\phi), \quad \tfrac{y}{h} = \tfrac{a}{b} - \tfrac{1}{b}(\sin 2\psi + 2\psi \sin\phi), \quad (\alpha\text{-lines}) \\ \tfrac{x-x_\beta}{h} &= -\tfrac{1}{b}(\cos 2\psi + 2\psi\cos\phi), \quad \tfrac{y}{h} = \tfrac{a}{b} - \tfrac{1}{b}(\sin 2\psi + 2\psi \sin\phi), \quad (\beta\text{-lines}). \end{aligned} \tag{4.4}$$

We denote by ψ_0 the value of ψ on the mid-plane $y=0$, and hence from (4.1) and (4.3)

$$a = \sin 2\psi_0 + 2\psi_0 \sin\phi. \tag{4.5}$$

Then the curve OA' is a segment of the α-line through O, defined parametrically as

$$\frac{x}{h} = \frac{1}{b}(\cos 2\psi - \cos 2\psi_0 - 2(\psi - \psi_0)\cos\phi), \quad \frac{y}{h} = -\frac{a}{b} + \frac{1}{b}(\sin 2\psi + 2\psi\sin\phi), \tag{4.6}$$

and OB' is the segment of the β-line through O, and is given by

$$\frac{x}{h} = \frac{1}{b}(\cos 2\psi - \cos 2\psi_0 + 2(\psi - \psi_0)\cos\phi), \quad \frac{y}{h} = -\frac{a}{b} + \frac{1}{b}(\sin 2\psi + 2\psi\sin\phi). \tag{4.7}$$

Similarly OA is a segments of the α-line in $x > 0$

$$\frac{x}{h} = -\frac{1}{b}(\cos 2\psi - \cos 2\psi_0 - 2(\psi - \psi_0)\cos\phi), \quad \frac{y}{h} = \frac{a}{b} - \frac{1}{b}(\sin 2\psi + 2\psi\sin\phi), \tag{4.8}$$

and OB is a segment of the β-line

$$\frac{x}{h} = -\frac{1}{b}(\cos 2\psi - \cos 2\psi_0 + 2(\psi - \psi_0)\cos\phi), \quad \frac{y}{h} = \frac{a}{b} - \frac{1}{b}(\sin 2\psi + 2\psi)\sin\phi). \tag{4.9}$$

In the regions OAB' and OBA' the stress is indeterminate, but must be in equilibrium and not violate the yield condition. This form of solution is suggested by the solution for pure compression of a layer by Hartmann [4] and Marshall [5], which in turn are extensions of Prandtl's solution [3]. The present solution reduces to that of Hartmann and Marshall when $a = 0$.

It follows from (2.6) that at $y = 0$

$$\sigma_{yy} = -q\,(\cos 2\psi_0 + \operatorname{co}\sec\phi)\,, \quad \sigma_{xy} = q\sin 2\psi_0. \tag{4.10}$$

We look for solutions in which the stress components are large in magnitude near $x = 0$ at the centre of the layer, and decay as $x \to \pm L$, and so we require $b > 0$. Also, for definiteness, we assume that $\sin 2\psi_0 > 0$, so that $a > 0$ and $\sigma_{xy} > 0$ at $y = 0$, and therefore the shear force on $y = 0$ acts in the positive x-direction. There are corresponding solutions with $\sin 2\psi_0 < 0$.

We also denote by ψ_1 the value of ψ on BC and $B'C'$, and by ψ_2 the value of ψ on AD and $A'D'$ so that

$$\begin{aligned} 2\psi_1 \sin\phi + \sin 2\psi_1 &= a - b, \\ 2\psi_2 \sin\phi + \sin 2\psi_2 &= a + b \end{aligned} \tag{4.11}$$

and hence

$$2a = 2(\psi_1 + \psi_2)\sin\phi + \sin 2\psi_1 + \sin 2\psi_2 = 2(2\psi_0 \sin\phi + \sin 2\psi_0),$$
$$2b = 2(\psi_2 - \psi_1)\sin\phi + \sin 2\psi_2 - \sin 2\psi_1. \qquad (4.12)$$

Since b is positive we must have $\psi_2 \geq \psi_1$. We also recall that in the case of pure compression considered by Hartmann [4] and Marshall [5], we have $a = 0$, and therefore in this case $\psi_0 = 0$ and $\psi_1 = -\psi_2$. In the case of steady simple shear, then $b = 0$ and $\psi_0 = \psi_1 = \psi_2 = \frac{1}{2}\phi + \frac{1}{4}\pi$. Hence we are concerned with values of ψ_0 in the range $0 \leq \psi_0 \leq \frac{1}{2}\phi + \frac{1}{4}\pi$.

We suppose that equal and opposite tractions act on the end faces CD and $C'D'$. Then, from (4.10), for equilibrium of the upper half of the layer we require

$$N = \int_{-L}^{L} q(\cos 2\psi_0 + \operatorname{cosec}\phi)\mathrm{d}x, \quad S = \int_{-L}^{L} q \sin 2\psi_0 \mathrm{d}x, \qquad (4.13)$$

and it follows from (4.1) and (4.3) that

$$N = 2Q(\cos 2\psi_0 + \operatorname{cosec}\phi)\exp(-\cos 2\psi_0 \tan\phi \sec\phi),$$
$$S = 2Q \sin 2\psi_0 \exp(-\cos 2\psi_0 \tan\phi \sec\phi) \qquad (4.14)$$

where

$$Q = C\frac{h}{b}\cos\phi \cot\phi \left\{1 - \exp\left(-\frac{bL}{h}\tan\phi \sec\phi\right)\right\}. \qquad (4.15)$$

If the layer is thin, in the sense that $h/L << 1$, then

$$Q \simeq C\frac{h}{b}\cos\phi \cot\phi. \qquad (4.16)$$

It follows from (4.14) that the layer has an apparent coefficient of friction $\tan\lambda_0$, where

$$\frac{S}{N} = \tan\lambda_0 = \frac{\sin 2\psi_0 \sin\phi}{\cos 2\psi_0 \sin\phi + 1}, \quad \text{and hence} \quad \sin\phi \sin(2\psi_0 - \lambda_0) = \sin\lambda_0. \qquad (4.17)$$

Alternatively, (4.17) can be regarded as determining the value of ψ_0 (and hence of a) that is required to support a force with normal and tangential components N and S applied to the surfaces of the layer. Since $\sin 2\psi_0$ is assumed to be positive, it follows that $\tan\lambda_0$ is positive. It follows from (4.17) that

$$0 \leq \lambda_0 < 2\psi_0 < \pi + \lambda_0.$$

Then in the case $a \geq 0, b \geq 0$ it follows from (4.12) that $\psi_2 \geq \psi_0 \geq \psi_1$.

A further boundary condition is required to determine the constant b. We assume Coulomb friction between the platens and the granular material, with angle of friction μ, so that at the surfaces $y = \pm h$, we have $|\sigma_{xy}| \leq -\sigma_{yy}\tan\mu$ and at yield $|\sigma_{xy}| = -\sigma_{yy}\tan\mu$. Since it has been assumed that the shear force on $y = h$ is in the positive x-direction, it follows that σ_{xy} is positive at points on the surface at which the critical shear stress is realized. We further denote

$$\tan\lambda_1 = \frac{\sin 2\psi_1 \sin\phi}{\cos 2\psi_1 + 1}, \quad \tan\lambda_2 = \frac{\sin 2\psi_2 \sin\phi}{\cos 2\psi_2 + 1}, \quad \sin\delta = \frac{\sin\mu}{\sin\phi}. \qquad (4.18)$$

Then if yield occurs on BC and $B'C'$, then $\lambda_1 = \mu$, and if yield occurs on AD and $A'D'$, then $\lambda_2 = \mu$. It follows from (4.18) that when $\lambda_1 = \mu$

$$2\psi_1 = \delta + \mu \qquad (4.19)$$

and when $\lambda_2 = \mu$

$$2\psi_2 = \delta + \mu. \tag{4.20}$$

Since $\psi_2 \geq \psi_1$, the critical shear stress is mobilized on AD and $A'D'$, when ψ_2 is given by (4.20). It then follows from (4.12) that

$$2\psi_1 \sin\phi + \sin 2\psi_1 = -(\mu + \delta)\sin\phi - \sin(\mu + \delta) + 2(2\psi_0 \sin\phi + \sin 2\psi_0). \tag{4.21}$$

For a specified value of ψ_0, and with ψ_2 given by the boundary condition (4.20), then (4.21) determines ψ_1.

An important special case is that of a perfectly rough surface, in which $\mu > \phi$ and yield occurs when the Coulomb shear stress $\sigma_{xy} = -\sigma_{yy} \tan\phi$ is mobilized on AD and $A'D'$. In this case the material shears on itself at these surfaces and in the above analysis we take $\mu = \phi$, so that $\sin\delta = 1$ and $\delta = \pi/2$, and

$$2\psi_2 = \phi + \frac{1}{2}\pi. \tag{4.22}$$

This case is adopted for the numerical illustrations given in Section 6.

To complete the analysis of the stress field it remains to determine the constant C, or equivalently Q. For this it is necessary to impose an additional condition on the confining stress in the granular layer, for example by specifying σ_{xx} at $x = \pm L$. Suppose, for example, that a horizontal confining pressure p_0 is applied at $x = \pm L$. Then from (2.6) and (4.3) it follows that

$$\begin{aligned} 2hp_0 &= -\int_{-h}^{h} \sigma_{xx}(L, y)\mathrm{d}y = -\int_{-h}^{h} q(L, y)\,(\cos 2\psi - \operatorname{cosec}\phi)\,\mathrm{d}y \\ &= -C\exp\left(-\frac{bL}{h}\tan\phi\sec\phi\right)\int_{-h}^{h}(\cos 2\psi - \operatorname{cosec}\phi)\exp\{-\cos 2\psi\tan\phi\sec\phi\}dy. \end{aligned}$$

Therefore, from (3.5)

$$\begin{aligned} 2hp_0 &= -2\frac{Ch}{b}\exp\left(-\frac{bL}{h}\tan\phi\sec\phi\right)\int_{\psi_2}^{\psi_1}(\cos 2\psi - \operatorname{cosec}\phi)\,(\cos 2\psi + \sin\phi) \\ &\qquad \times \exp\{-\cos 2\psi\tan\phi\sec\phi\}\,\mathrm{d}\psi \\ &= 2\frac{Ch}{b}\exp\left(-\frac{bL}{h}\tan\phi\sec\phi\right)\int_{\psi_2}^{\psi_1}\left(\sin^2 2\psi + \cos 2\psi\cot\phi\cos\phi\right) \\ &\qquad \times \exp\{-\cos 2\psi\tan\phi\sec\phi\}\,\mathrm{d}\psi \\ &= \frac{Ch}{b}\exp\left(-\frac{bL}{h}\tan\phi\sec\phi\right)\cot\phi\cos\phi[\sin 2\psi\exp\{-\cos 2\psi\tan\phi\sec\phi\}]_{\psi_2}^{\psi_1}. \end{aligned} \tag{4.23}$$

Hence, from (4.15)

$$Q = \frac{2hp_0\left\{\exp\left(b\frac{L}{h}\tan\phi\sec\phi\right) - 1\right\}}{[\exp\{-\cos 2\psi_1\tan\phi\sec\phi\}\sin 2\psi_1 - \exp\{-\cos 2\psi_2\tan\phi\sec\phi\}\sin 2\psi_2]}, \tag{4.24}$$

or, if $L >> h$

$$Q \approx \frac{2hp_0\exp\left(b\frac{L}{h}\tan\phi\sec\phi\right)}{[\exp\{-\cos 2\psi_1\tan\phi\sec\phi\}\sin 2\psi_1 - \exp\{-\cos 2\psi_2\tan\phi\sec\phi\}\sin 2\psi_2]}. \tag{4.25}$$

As noted above, OAB' and $OA'B$ are transition zones bounded by the four segments of characteristics through the origin, as illustrated schematically in Figure 2. The stress in this zone is indeterminate, but must conform to the condition (2.5). There are also regions adjacent to the ends $x = \pm L$ in which the stress depends on the details of the boundary conditions at these surfaces. The analysis by Hill, Lee and Tupper [18] of a related problem in metal plasticity suggests that the required superposed stress field decays rapidly with distance from the ends.

5. Compression and shear of a granular layer. Velocity field

We assume that the origin O is fixed and that the upper and lower platens can move as rigid bodies with velocities that are of equal magnitude but opposite in direction. Let us first consider the case that the deformation is isochoric, so that, from (3.13),

$$u = U\left\{\frac{x}{h} - \frac{2}{b}(\cos 2\psi - \cos 2\psi_0)\right\}, \quad v = -U\frac{y}{h} \quad \text{in} \quad OA'D'C'B',$$
$$u = U\left\{\frac{x}{h} + \frac{2}{b}(\cos 2\psi - \cos 2\psi_0)\right\}, \quad v = -U\frac{y}{h} \quad \text{in} \quad OADCB, \tag{5.1}$$

where the constant U_0 has been chosen so that $u = 0$ at the origin. The deformation (5.1) comprises a pure shear deformation, described by

$$u = Ux/h, \quad v = -Uy/h$$

on which is superposed a non-uniform shearing deformation

$$\begin{aligned} u &= -\tfrac{2U}{b}(\cos 2\psi - \cos 2\psi_0), \quad v = 0 \quad \text{in} \quad OA'D'C'B', \\ u &= \tfrac{2U}{b}(\cos 2\psi - \cos 2\psi_0), \quad v = 0 \quad \text{in} \quad OADCB. \end{aligned} \tag{5.2}$$

Because of the dependence on x in (5.1), this motion implies slip between the material and the platens at $y = \pm h$ such that the magnitude of the slip increases linearly with x. For a thin granular layer this seems unrealistic, because it is clear from the analysis of Section 4 that large compressive stresses and large frictional forces are generated at the surfaces $y = \pm h$, especially in the neighbourhood of $x = 0$. We therefore propose that it is more realistic to assume that such slip does not occur, and that the consequent change in volume of the granular layer is accommodated by elastic compression, consolidation, or crushing of the granules. For example the forces applied may result in the crushing strength of the material being exceeded, and then the dilatation-rate Δ depends on the history of the applied pressure, while the surface friction is sufficient to prevent surface slip. In such a case, the theory of Spencer and Kingston [15], with solutions of the form (3.18), is appropriate. Hence we assume

$$\begin{aligned} u &= -\tfrac{U}{b}(\cos 2\psi - \cos 2\psi_0), \quad v = -U\tfrac{y}{h}, \quad \text{in} \quad OA'D'C'B' \\ u &= \tfrac{U}{b}(\cos 2\psi - \cos 2\psi_0), \quad v = -U\tfrac{y}{h}, \quad \text{in} \quad OADCB. \end{aligned} \tag{5.3}$$

Hence if $u = \pm\gamma h$ at $y = \pm h$

$$\gamma h = \frac{U}{b}(\cos 2\psi_1 - \cos 2\psi_0), \quad -\gamma h = \frac{U}{b}(\cos 2\psi_2 - \cos 2\psi_0) \tag{5.4}$$

and therefore the mean shear strain-rate γ is, from (4.12)

$$\gamma = \frac{U(\cos 2\psi_1 - \cos 2\psi_2)}{2hb} = \frac{U(\cos 2\psi_1 - \cos 2\psi_2)}{h\left\{2(\psi_2 - \psi_1)\sin\phi + \sin 2\psi_2 - \sin 2\psi_1\right\}}. \tag{5.5}$$

We note that in the limit $\psi_1 \to \psi_2$, which is the case of uniform steady shear stress

$$\gamma \to \frac{U \sin 2\psi_2}{h(\cos 2\psi_2 + \sin\phi)} \tag{5.6}$$

and in particular that γ is unbounded in the limit $2\psi_2 = 2\psi_1 \to \pi/2 + \phi$, which is the case of steady simple shearing deformations. It is also straightforward to show that if $2\psi_2$ and $2\psi_1$ are close to the limit value $\pi/2 + \phi$, for example

$$2\psi_2 = \frac{\pi}{2} + \phi - \varepsilon_2, \quad 2\psi_1 = \frac{\pi}{2} + \phi - \varepsilon_1, \tag{5.7}$$

with $\varepsilon_1, \varepsilon_2 << 1$, then

$$\gamma = \frac{2U}{h(\varepsilon_1 + \varepsilon_2)} + O(1), \tag{5.8}$$

and so the shear-rate increases rapidly as $2\psi_2$ and $2\psi_1$ tend to $\pi/2 + \phi$.

The transition regions $OA'B$ and OAB' in the vicinity of $x = 0$ move as rigid bodies with speed U, in the positive y-direction for $y < 0$ and in the negative y-direction for $y > 0$.

6. Numerical results and discussion

For the purposes of the calculations presented in this section , we assume that the platen surfaces are perfectly rough, so that $\mu \geq \phi$, and $2\psi_2 = \phi + \frac{1}{2}\pi$, and assign the typical value $\pi/6$ to ϕ. With these values Figure 3 shows the variation of ψ_1 with ψ_0, where from (4.12) and (4.20)

$$2\psi_1 \sin\phi + \sin 2\psi_1 = 2(2\psi_0 \sin\phi + \sin 2\psi_0) - \left(\phi + \frac{1}{2}\pi\right) \sin\phi - \cos\phi. \tag{6.1}$$

We recall that ψ_0 varies from $\psi_0 = 0$, for the case of compression with no resultant shear force, to $\psi_0 = \frac{1}{2}\phi + \frac{1}{4}\pi$ for the case of steady simple shear. Correspondingly, ψ_1 varies from $\psi_1 = -\frac{1}{2}\phi - \frac{1}{4}\pi$ (so that $\psi_1 = -\psi_2$) for pure compression, to $\psi_1 = \frac{1}{2}\phi + \frac{1}{4}\pi$ (or $\psi_1 = \psi_0 = \psi_2$) for steady simple shear.

In Figure 4 we show the variation of $S/N = \tan\lambda_0$ (where λ_0 is given by (4.17)) through this range of values of ψ_0, again for the case $\phi = \pi/6$.

Figure 5 shows the variation of the parameters a and b with ψ_0. These parameters characterize the stress and velocity fields and are defined by (4.12). Then, for given values of ϕ, h, L

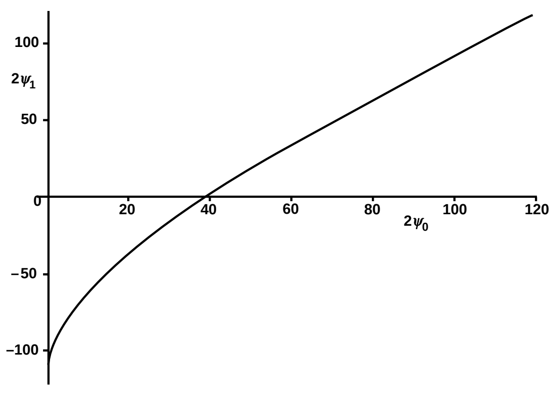

Figure 3. Variation of the angle ψ_1 with the angle ψ_0.

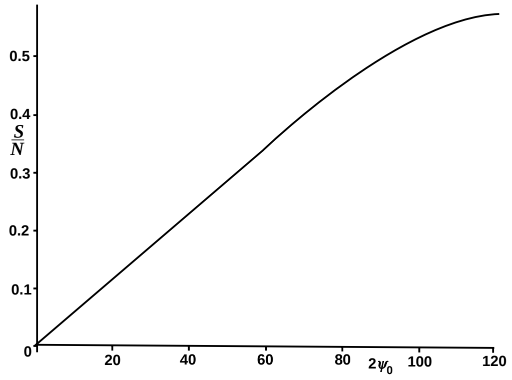

Figure 4. Variation of the ratio S/N of shear force to normal force with the angle ψ_0.

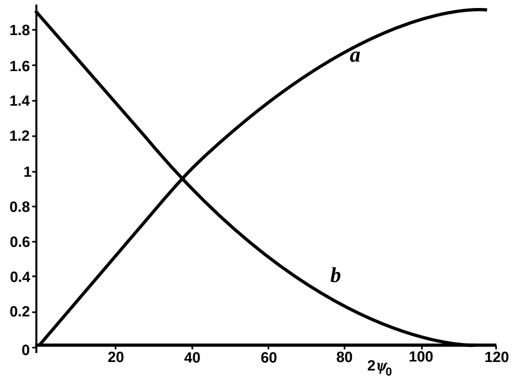

Figure 5. Variation of the parameters a and b with the angle ψ_0.

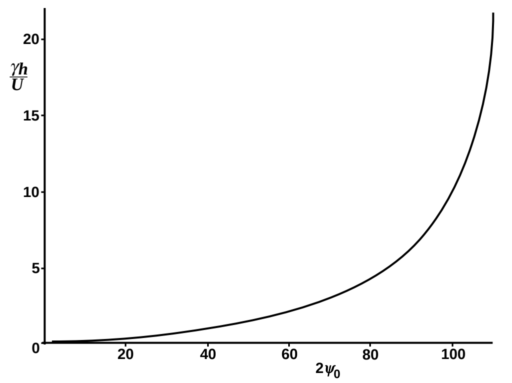

Figure 6. Variation of the shear-rate γ with the angle ψ_0.

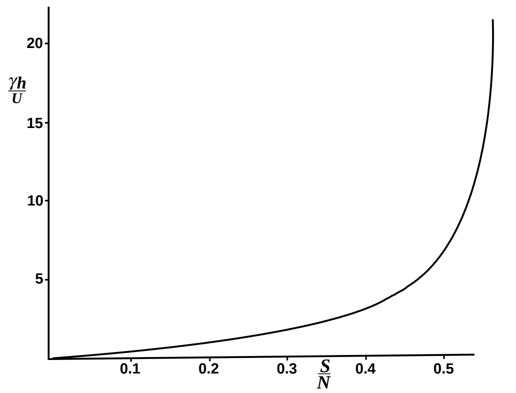

Figure 7. Variation of the shear-rate γ with the ratio S/N of shear force to normal force.

and p_0, Q is determined by (4.22) or (4.23). The values of the resultant normal force N and the resultant shear force S for a given value of ψ_0 then follow from (4.14).

In Figure 6 we show the variation of $h\gamma/U$ with ψ_0, as given by (5.5). Here γ is the average shear strain-rate and U/h is the direct compressive strain-rate. Figure 7 shows how S/N varies with $h\gamma/U$.

Some points of interest arise from these results. It is clear that over most of the range the shear force component has only a minor effect on the deformation and the compressive component predominates. The shearing component of the deformation only becomes significant when S/N is close to $\tan\phi$.

It was pointed out in [8,9] that the steady simple shearing deformation with $\sigma_{xy} = -\sigma_{yy}\tan\phi$ and $\psi_0 = \frac{1}{4}\pi + \frac{1}{2}\phi$ is unstable, in the sense that, if the traction boundary conditions permit it, shear flow of the material is accompanied by a decreasing shear stress. In this paper, we have assumed that S and N are specified, and therefore ψ_0 is fixed. In this case the steady solution is the only possible continuous solution, but the possibility of strain localization or shear band formation cannot be excluded. It appears from the analysis of this paper that if the shear is accompanied by compression, which is our general case, then the compression will have a stabilizing effect. This is because if $U > 0$ then h decreases after yield occurs, and therefore, from (4.22), if the other parameters are fixed, Q increases after yield takes place, and so the deformation takes place under increasing load. In principle, if the post-yield behaviour of S and N is specified, then the post-yield stress-strain relation may be deduced from (4.14), (4.22), (5.5) and the relation $U = -\mathrm{d}h/\mathrm{d}t$. An unresolved problem is that the equations of the double-shearing theory are known to be mathematically ill-posed, but the present solution does not seem to shed any light on this difficulty.

Acknowledgement

A large part of this paper was written at the Isaac Newton Institute for Mathematical Sciences, Cambridge during the author's participation in the Institute's programme on 'Granular and Particle-laden Flows'.

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